

1D multicomponent Fermions with delta function interaction in strong and weak coupling limits: κ -component Fermi gas

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We derive the first few terms of the asymptotic expansion of the Fredholm equations for one-dimensional κ -component fermions with repulsive and with attractive delta-function interaction in strong and weak coupling regimes. We thus obtain a highly accurate result for the ground state energy of a multicomponent Fermi gas with polarization for these regimes. This result provides a unified description of the ground state properties of the Fermi gas with higher spin symmetries. However, in contrast to the two-component Fermi gas, there does not exist a mapping that can unify the two sets of Fredholm equations as the interacting strength vanishes. Moreover, we find that the local pair correlation functions shed light on the quantum statistic effects of the κ -component interacting fermions. For the balanced spin case with repulsive interaction, the ground state energy obtained confirms Yang and You's result [Chin. Phys. Lett. **28**, 020503 (2011)] that the energy per particle as $\kappa \rightarrow \infty$ is the same as for spinless Bosons.

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I. INTRODUCTION

The recently established experimental control over the effective spin-spin interaction between atoms is opening up new avenues for studying spin effects in low-dimensional atomic quantum gases with higher spin symmetries. Fermionic alkaline-earth atoms display an exact $SU(\kappa)$ spin symmetry with $\kappa = 2I + 1$ where I is the nuclear spin [1]. For example, a recent experiment [2] for ^{171}Yb dramatically realised the model of fermionic atoms with $SU(2) \otimes SU(6)$ symmetry where electron spin decouples from its nuclear spin $I = 5/2$. Such fermionic systems with enlarged $SU(\kappa)$ spin symmetry are expected to display a remarkable diversity of new quantum phases and quantum critical phenomena due to the rich linear and nonlinear Zeeman effects. The study of low-dimensional cold atomic Fermi gases with higher pseudo-spin symmetries has become a new frontier in cold atom physics.

One-dimensional (1D) quantum Fermi gases with delta-function interaction are important exactly solvable quantum many-body systems and have had tremendous impact in quantum statistical mechanics. The spin-1/2 Fermi gas with arbitrary polarization was solved long ago by Yang [3] using the Bethe ansatz (BA) hypothesis. Sutherland [4] generalized the result of the spin-1/2 Fermi gas to 1D multi-component Fermi gas in 1968. The study of multicomponent attractive Fermi gases was initiated by Yang [5] and by Takahashi [6]. Using Yang and Yang's method [7] for the boson case, Takahashi [8] and Lai [9] derived the thermodynamic Bethe ansatz (TBA) equations for spin-1/2 fermions. In the same fashion, Schlottmann [10] derived the TBA equations for $SU(\kappa)$ fermions with repulsive and attractive interactions respectively. Recently, the thermodynamics of the multi-component Fermi gas was obtained by solving the TBA

equations in [12]. These models with enlarged spin symmetries have received a renewed interest in cold atom physics [11, 13–15].

In this communication, we consider 1D κ -component fermions with repulsive and with attractive delta-function interactions. Although the model was solved long ago by Sutherland [4], solutions of the Fredholm equations for the model are far from being thoroughly investigated except for the two-component Fermi gas [16]. From a theoretical point of view, finding a general form of the solutions to the Fredholm equations for multicomponent Fermi gases with higher spin symmetry imposes a number of challenges.

In the present paper, using the method which we recently developed in the analytical study of the 1D two-component Fermi gas [16], we approximately solve the Fredholm equations of the 1D κ -component fermions with polarization for the A) strongly repulsive regime; B) weakly repulsive regime; C) weakly attractive regime; and D) strongly attractive regime. We thus obtain the ground state energy of the κ -component Fermi gas with polarization that provides a fundamental understanding of the ground state properties, such as phase diagram, magnetism and quantum statistical effects. In contrast to the two-component Fermi gas [16], the two sets of Fredholm equations for the multi-component Fermi gas with weakly repulsive and attractive interactions can not be unified by the density mapping as the interacting strength vanishes. The multiple spin degrees of freedom impose subtle intricacies of quantum statistics in the ground state properties of the systems. We further study the local pair correlation functions for the κ -component interacting fermions. We find that the local pair correlation as $\kappa \rightarrow \infty$ is the same as for spinless bosons. This result is consistent with Yang and You's finding [19] that the energy per particle as $\kappa \rightarrow \infty$ is the same as for

spinless Bosons.

II. THE FREDHOLM EQUATIONS

The Hamiltonian for the 1D N -body problem is [3, 4]

$$H = -\frac{\hbar^2}{2m} \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + g_{1D} \sum_{1 \leq i < j \leq N} \delta(x_i - x_j). \quad (1)$$

It describes N fermions of the same mass m confined to a 1D system of length L interacting via a δ -function potential. There are κ possible hyperfine states $|1\rangle, |2\rangle, \dots, |\kappa\rangle$ that the fermions can occupy. For an irreducible representation $[\kappa^{N_\kappa}, (\kappa-1)^{N_{\kappa-1}}, \dots, 2^{N_2}, 1^{N_1}]$, the Young diagram has κ columns with the quantum numbers $N_i = N^i - N^{i+1}$, here the N^i is the numbers of fermions at the i -th hyperfine levels such that $N^1 \geq N^2 \geq \dots \geq N^\kappa$. This system has $SU(\kappa)$ spin symmetry and $U(1)$ charge symmetry. The coupling constant g_{1D} can be expressed in terms of the interaction strength $c = -2/a_{1D}$ as $g_{1D} = \hbar^2 c/m$ where a_{1D} is the effective 1D scattering length. Here $c > 0$ for repulsive fermions, and $c < 0$ for attractive fermions.

The energy eigenspectrum is given in terms of the quasimomenta $\{k_i\}$ of the fermions via $E = \sum_{j=1}^N k_j^2$, which in terms of the function $e_b(x) = (x + ibc/2)/(x - ibc/2)$ satisfy the BA equations [4-6]

$$\begin{aligned} \exp(ik_i L) &= \prod_{\alpha=1}^{M_1} e_1 \left(k_i - \lambda_\alpha^{(1)} \right), \\ \prod_{\beta=1}^{M_{\ell-1}} e_1 \left(\lambda_\alpha^{(\ell)} - \lambda_\beta^{(\ell-1)} \right) \\ &= - \prod_{\eta=1}^{M_\ell} e_2 \left(\lambda_\alpha^{(\ell)} - \lambda_\eta^{(\ell)} \right) \prod_{\delta=1}^{M_{\ell+1}} e_{-1} \left(\lambda_\alpha^{(\ell)} - \lambda_\delta^{(\ell+1)} \right) \end{aligned} \quad (2)$$

Where $i = 1, \dots, N$, $\alpha = 1, \dots, M_\ell$ and the parameters $\{\lambda_\alpha^{(\ell)}\}$ with $\ell = 1, 2, \dots, \kappa - 1$ are the spin rapidities while we denote $\lambda_\alpha^{(0)} = k_\alpha$, $\lambda_\alpha^{(\kappa)} = 0$ and $c' = c/2$. The BA quantum numbers $M_i = \sum_{j=i}^{\kappa-1} (j - i + 1) N_{j+1}$ with $M_\kappa = 0$ in the above BA equations (2).

A. Repulsive regime

The fundamental physics of the model are determined by the set of transcendental BA equations (2) which can be transformed to generalised Fredholm equations in the thermodynamic limit. The Fredholm equations for repulsive and attractive regimes are significantly different. From the BA equations (2), the quasimomenta $\{k_i\}$ are real, but all $\{\lambda_\alpha^{(\ell)}\}$ are real only for the ground state. For

the ground state, the generalized Fredholm equations for $c > 0$ are given by [4-6]

$$r_0(k) = \beta_0 + \int_{-B_1}^{B_1} K_1(k - k') r_1(k') dk', \quad (3)$$

$$\begin{aligned} r_m(k) &= \int_{-B_{m-1}}^{B_{m-1}} K_1(k - \lambda) r_{m-1}(\lambda) d\lambda \\ &\quad - \int_{-B_m}^{B_m} K_2(k - \lambda) r_m(\lambda) d\lambda \\ &\quad + \int_{-B_{m+1}}^{B_{m+1}} K_1(k - \lambda) r_{m+1}(\lambda) d\lambda, \end{aligned} \quad (4)$$

where $1 \leq m \leq \kappa - 1$ and $\beta_0 = 1/(2\pi)$, $r_0(k)$ is the particle quasimomentum distribution function whereas $r_m(k)$ with $m \geq 1$ are the distribution functions for the $\kappa - 1$ spin rapidities. The kernel function $K_\ell(k)$ is defined as

$$K_\ell(k) = \frac{1}{2\pi} \frac{\ell c}{(\ell c/2)^2 + k^2}, \quad (5)$$

here $c > 0$ for repulsive interaction and $c < 0$ for attractive interaction. Following the method used for the two-component Fermi gas [16], we rewrite the Fredholm equations (4) as

$$\begin{aligned} r_m(k) &= \beta_0 - \sum_{s=0}^{m-1} \int_{|\lambda| > B_s} K_{m-s}(k - \lambda) r_s(\lambda) d\lambda \\ &\quad + \int_{-B_{m+1}}^{B_{m+1}} K_1(k - \lambda) r_{m+1}(\lambda) d\lambda, \end{aligned} \quad (6)$$

where $0 \leq m \leq \kappa - 1$. The associating integration boundaries B_m are determined by the conditions

$$m_\ell \equiv \frac{M_\ell}{L} = \int_{-B_\ell}^{B_\ell} r_\ell(k) dk, \quad 0 \leq \ell \leq \kappa - 1, \quad (7)$$

where $M_0 = N$ is the total number of fermions, $N^\ell = M_{\ell-1} - M_\ell$ is the number of fermions in the ℓ -th hyperfine state. Here $M_\kappa = 0$. The ground state energy E per unit length is given by

$$E = \int_{-B_0}^{B_0} k^2 r_0(k) dk. \quad (8)$$

The model has $SU(\kappa)$ symmetry in spin sector and $U(1)$ symmetry in charge sector. Therefore, the quantum numbers of each spin states are conserved. Thus the system has κ chemical potentials $\{\mu_\ell\}$ in regard to these conserved numbers N^ℓ . The ground state energy (8) is a smooth function of the densities of $n^\ell = N^\ell/L$ with $\ell = 1, 2, \dots, \kappa$ for unbalanced case. In the grand canonical ensemble, we can also get the chemical potentials μ_ℓ via $\mu_\ell = \partial E / \partial n^\ell$.

B. Attractive regime

In the attractive regime, it is found that complex string solutions of k_j also satisfy the BA equations [5, 6, 17]. The quasimomenta k_j may appear as bound states of m -atom up to length $2, \dots, \kappa$. A bound state in quasimomentum space of length m takes on the form

$$k_\alpha^{m,j} = \lambda_\alpha^{(m-1)} + i(m+1-2j)|c'| + O(\exp(-\delta L)), \quad (9)$$

where $j = 1, \dots, m$. The number of bound states with length $1 \leq m \leq \kappa$ is denoted as N_m . Its real part is $\lambda_\alpha^{(m-1)}$. A k_α bound state of m -atom will be accompanied by a $\lambda_\alpha^{(1)}$ string of length $m-1$, a $\lambda_\alpha^{(2)}$ string of length $m-2$ and so on until a $\lambda_\alpha^{(m-1)}$ string of length 1. Each accompanying string state in $\lambda^{(1)}$ -space, $\lambda^{(2)}$ -space, \dots , $\lambda^{(m-1)}$ -space will share the same real part $\lambda_\alpha^{(m-1)}$, see [5, 6, 17] and the second reference in [12]. The unpaired atoms have real quasimomenta k_i 's.

From these quasimomentum bound states, the Fredholm equations for the model with an attractive interaction are given by [5, 6]

$$\begin{aligned} \rho_m(\lambda) = & m\beta_0 + \sum_{r=1}^{m-1} \sum_{s=r}^{\kappa} \int_{-Q_s}^{Q_s} K_{s+m-2r}(\lambda - \Lambda) \rho_s(\Lambda) d\Lambda \\ & + \sum_{s=m+1}^{\kappa} \int_{-Q_s}^{Q_s} K_{s-m}(\lambda - \Lambda) \rho_s(\Lambda) d\Lambda, \end{aligned} \quad (10)$$

where $\rho_1(k)$ is the density distribution function of single fermions, whereas $\rho_m(k)$ is the density distribution function for the bound state of m -atom with $1 < m \leq \kappa$. The total number of fermions is given by $N = \sum_{m=1}^{\kappa} mN_m$. The integration boundaries Q_m , characterizing the Fermi points in each Fermi sea, are determined by

$$n_m \equiv \frac{N_m}{L} = \int_{-Q_m}^{Q_m} \rho_m(k) dk. \quad (11)$$

The ground state energy per unit length is given by

$$E = \sum_{m=1}^{\kappa} \int_{-Q_m}^{Q_m} \left(mk^2 - \frac{m(m^2-1)}{12} c^2 \right) \rho_m(k) dk. \quad (12)$$

In the attractive regime, for convenience, we can define the effective chemical potentials for the cluster bound states $\mu_m = \mu + H_m/m + (m^2-1)c^2/12$ with $m = 1, \dots, \kappa$, where H_m is the effective magnetic field (or Zeeman splitting parameter) for the bound state of m -atom, see [12]. The Zeeman energy per unit length can be written as $E_z = \sum_{m=1}^{N-1} H_m N_m / L$. In this regime, the effective chemical potentials for the bound states of different sizes can be derived from the energies of the ground state, i.e.

$$\mu_m = \frac{1}{m} \frac{\partial}{\partial n_m} \sum_{s=1}^{\kappa} \int_{-Q_s}^{Q_s} s k^2 \rho_s(k) dk. \quad (13)$$

Following [12], the full phase diagrams of the model can be determined by the field-energy transfer relations

$$H_m = \frac{1}{12} m(\kappa^2 - m^2) c^2 + m(\mu_m - \mu_\kappa) \quad (14)$$

with $m = 1, 2, \dots, \kappa$. In the next section, we shall derive the explicit form of the ground state energy that can give highly accurate effective chemical potentials μ_m to determine magnetism and full phase diagrams of the model.

Similarly, the Fredholm equations can be rewritten as

$$\begin{aligned} \rho_m(\lambda) = & \beta_0 - \int_{|\Lambda| > Q_{m-1}} K_1(\lambda - \Lambda) \rho_{m-1}(\Lambda) d\Lambda \\ & + \sum_{s=m+1}^{\kappa} \int_{-Q_s}^{Q_s} K_{s-m}(\lambda - \Lambda) \rho_s(\Lambda) d\Lambda. \end{aligned} \quad (15)$$

We see that there is a particular mapping between the two sets of Fredholm equations (6) and (15), i.e.

$$\begin{aligned} \rho_m & \rightarrow r_{\kappa-m}, & Q_m & \rightarrow B_{\kappa-m}, \\ \int_{-Q_m}^{Q_m} & \rightarrow \int_{-\infty}^{\infty} - \int_{-B_{\kappa-m}}^{B_{\kappa-m}}, & c & \rightarrow -c. \end{aligned} \quad (16)$$

This connection is useful in the analysis of the symmetric structure between the two sides. However, we find that the Fredholm equations (6) for the repulsive regime and (15) for the attractive regime do not preserve the mapping which exists between the two sets of the Fredholm equations for the two-component Fermi gas [16]. This leads to particular intricacies in the analytical behaviour of the ground state energy as the interaction strength vanishes. In the present paper, we mainly concentrate on the solutions of the two sets of the Fredholm equations for multicomponent Fermi gas with attractive and repulsive interactions.

III. ASYMPTOTIC SOLUTIONS OF THE FREDHOLM EQUATIONS

A. Strong repulsion

For the balanced case, i.e. $M_m - M_{m+1} = N/\kappa$, the following theorem shows that all integration boundaries $B_m \rightarrow \infty$ except for $m = 0$, namely, there are no finite Fermi points (without chemical biases between different spin states).

Theorem: For the κ -component fermion system with the repulsive delta-function interaction, $B_m \rightarrow \infty$ for $m = 1, \dots, \kappa - 1$ if the relation

$$M_{m-1} - M_m = M_m - M_{m+1} \quad (17)$$

holds, and vice versa.

Proof: Integrating both sides of the Fredholm equations (4) with infinite boundary (where $m \neq 0$), we obtain

$$\int_{-\infty}^{\infty} r_m(k) dk = \frac{M_{m-1}}{L} - \frac{M_m}{L} + \frac{M_{m+1}}{L}. \quad (18)$$

If the condition (17) holds, thus we have

$$\int_{-\infty}^{\infty} r_m(k) dk = \frac{M_m}{L} \quad (19)$$

where the integration boundary $B_m = \infty$ is inferred. Conversely, if B_m with $m \geq 1$ tend to infinite, from (4) and (7) we have

$$\frac{M_m}{L} = \int_{-\infty}^{\infty} r_m(k) dk = \frac{M_{m-1}}{L} - \frac{M_m}{L} + \frac{M_{m+1}}{L}. \quad \diamond$$

For the balanced case, the integration boundaries B_m with $m \geq 1$ are infinitely large. The fermi momentum B_0 is always finite. For convenience, we introduce the notation $r_0(k) = r_{0\text{in}}(k) + r_{0\text{out}}(k)$ where

$$\begin{aligned} r_{0\text{in}}(k) &= \begin{cases} r_0(k) & \text{for } |k| \leq B_0 \\ 0 & \text{for } |k| > B_0 \end{cases} \\ r_{0\text{out}}(k) &= \begin{cases} 0 & \text{for } |k| \leq B_0 \\ r_0(k) & \text{for } |k| > B_0 \end{cases}. \end{aligned}$$

Taking the Fourier transformation with the Fredholm equations (4) for the balanced case, we obtain the following relations

$$\begin{aligned} \tilde{r}_m(\omega) &= \frac{F_{m+1}(\omega)}{F_m(\omega)} \tilde{r}_{m-1}(\omega), \quad \tilde{r}_1(\omega) = \frac{F_2(\omega)}{F_1(\omega)} \tilde{r}_{0\text{in}}(\omega), \\ F_m(\omega) &= \sum_{s=0}^{\kappa-m} e^{-(\kappa-m+2s)c|\omega|/2}, \end{aligned} \quad (20)$$

where $2 \leq m \leq \kappa - 1$ and $F_\kappa = 1$. We denote the Fourier transform $\hat{F}[r_m(k)] = \tilde{r}_m(\omega)$. From the Fredholm equation (3), we see that a closed form of the distribution function $r_1(k)$ is essential for the calculation of the ground state energy. After a straightforward calculation with the relations (20), we obtain the closed form

$$\tilde{r}_m(\omega) = \frac{\tilde{r}_{0\text{in}}(\omega) \sinh \left[\frac{1}{2}(\kappa - m)|\omega|c \right]}{\sinh \left[\frac{1}{2}\kappa|\omega|c \right]} \quad (21)$$

that gives the distribution function

$$r_m(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{r}_m(\omega) e^{-i\omega\lambda} d\omega. \quad (22)$$

In the above equations $\tilde{r}_{0\text{in}}(\omega)$ can be determined from the following relation

$$\begin{aligned} I(k) &= \int_{-B_0}^{B_0} K_1(k - \lambda) r_0(\lambda) d\lambda \\ &= \int_{-\infty}^{\infty} K_1(k - \lambda) r_{0\text{in}}(\lambda) d\lambda, \end{aligned} \quad (23)$$

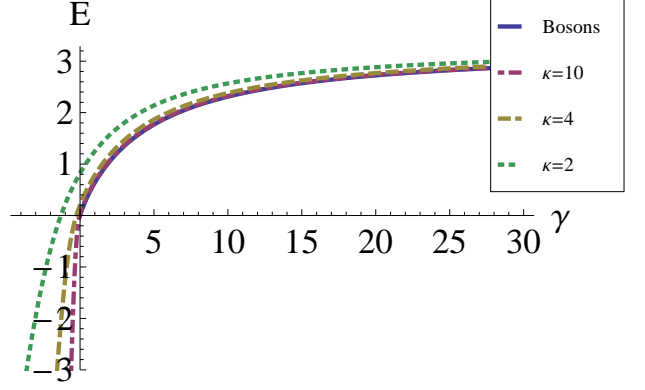


FIG. 1: The ground state energy per length vs $\gamma = cL/N$ in natural units of $2m = \hbar = 1$ for the balanced multicomponent Fermi gas: $\kappa = 2, 4, 10$ and for spinless Bose gas in the whole interacting regime. For the repulsive regime, the discrepancy between the energies of the $\kappa = 10$ Fermi gas and the spinless Bose gas is negligible. As $\kappa \rightarrow \infty$, the ground state energy of the balanced repulsive Fermi gas fully coincides with that of the spinless Bose gas [19].

where their Fourier transform reads $\tilde{I}(\omega) = e^{-c|\omega|/2} \tilde{r}_{0\text{in}}(\omega)$. The ground state energy per unit length (8) can be calculated either numerically or asymptotically. In Fig. 1, we plot the ground state energy of 1D balanced multicomponent Fermi gas for $\kappa = 2, 4, 10$ and for spinless Bose gas [20]. We see that as $\kappa \rightarrow \infty$, the ground state energy of the balanced κ -component gas with repulsion coincides with that of the 1D spinless Bose gas [19]. In order to capture the statistical nature for this connection, we shall calculate the first few terms of the expansion of the ground state energy.

The strong repulsion condition, i.e., $cL/N \gg 1$, naturally gives $c \gg B_0$, where B_0 is proportional to the Fermi velocity $k_F = n\pi$. Following the method used in [16], we take a Taylor expansion with respect to λ for the kernel function $K_1(k - \lambda)$ in eq. (23). Using the relations (7), (8), (22) and (23), we find

$$\begin{aligned} \tilde{r}_{0\text{in}}(\omega) &\approx n - \frac{E\omega^2}{2}, \\ \tilde{r}_1(\omega) &\approx \frac{F_2(\omega)}{F_1(\omega)} \left[n - \frac{E\omega^2}{2} \right]. \end{aligned} \quad (24)$$

In the above equations, we used the following formulas

$$\begin{aligned} \hat{F} \left[(c^2/4 + k^2)^{-2} \right] &= 2\pi c^{-3} e^{-\frac{c|\omega|}{2}} [c|\omega| + 2], \\ \hat{F} \left[(c^2/4 + k^2)^{-3} \right] &= \pi c^{-5} e^{-\frac{c|\omega|}{2}} [c^2|\omega|^2 + 6c|\omega| + 12]. \end{aligned}$$

Substituting (24) into the Fredholm equation (3), we obtain an asymptotic form of the density distribution function

$$r_0(k) = \frac{1}{2\pi} + \frac{NY_0(k)}{2\pi L} - \frac{EY_2(k)}{4\pi} + O(c^{-4}), \quad (25)$$

where the function

$$Y_\alpha(k) \approx \int_{-\infty}^{\infty} \frac{e^{i\omega k} e^{-\frac{c|\omega|}{2}} \omega^\alpha F_2(\omega)}{F_1(\omega)} d\omega. \quad (26)$$

After some algebra, we obtain the two functions used in (25)

$$\begin{aligned} Y_0(k) &= \frac{2Z_1}{c} - \frac{2Z_3 k^2}{c^3} + O(c^{-4}) \\ Y_2(k) &= \frac{4Z_3}{c^3} + O(c^{-4}), \end{aligned}$$

with

$$\begin{aligned} Z_1 &= -\frac{1}{\kappa} \left[\psi\left(\frac{1}{\kappa}\right) + C \right], \\ Z_3 &= \kappa^{-3} \left[\zeta\left(3, \frac{1}{\kappa}\right) - \zeta(3) \right]. \end{aligned} \quad (27)$$

Here $\zeta(z, q)$ and $\zeta(z)$ are the Riemann zeta functions, $\psi(p)$ denotes the Euler psi function, C denotes the Euler constant. When $\kappa = 2$ we have $Z_1 = \ln 2$, $Z_3 = (3/4)\zeta(3)$ that are consistent with the result given in [16].

Substituting (25) into (7) and (8) and solving by iteration, the Fermi boundary B_0 and the ground state energy of balanced κ -component Fermi gas with a strong repulsion are given explicitly (up to order $O(c^{-3})$)

$$\begin{aligned} B_0 &\approx n\pi \left\{ 1 - \frac{2Z_1}{\gamma} + \frac{4Z_1^2}{\gamma^2} - \frac{8Z_1^3}{\gamma^3} + \frac{4Z_3\pi^2}{3\gamma^3} \right\}, \quad (28) \\ E &\approx \frac{n^3\pi^2}{3} \left\{ 1 - \frac{4Z_1}{\gamma} + \frac{12Z_1^2}{\gamma^2} - \frac{32}{\gamma^3} \left(Z_1^3 - \frac{Z_3\pi^2}{15} \right) \right\} \quad (29) \end{aligned}$$

where the dimensionless interaction strength $\gamma = c/n$. The ground state energy (29) with $\kappa = 2$ reduces to the result given in [16]. In view of the quasi-momentum distribution $r_0(k)$ (25), the system can be taken as an ideal gas with a less exclusive fractional statistics than the Fermi statistics. It is interesting to see that the ground state energy (29) reduces to the energy of the spinless Bose gas (see (10) in [18]) as $\kappa \rightarrow \infty$, also see [19]. Here we find that $\lim_{\kappa \rightarrow \infty} Z_1 = \lim_{\kappa \rightarrow \infty} Z_3 = 1$. A significant interpolation between Fermions and Bosons can be conceived from the parameters Z_1 and Z_2 in (29) that encodes the quantum statistical and dynamical effects. In Fig. 2, we see a good agreement between the asymptotic expansion result (29) and numerical result obtained from the Fredholm equation (3) with the density distribution (22).

In general, the boundaries B_m with $m > 1$ are hard to be estimated for the imbalanced case in a strong repulsive regime. However, for strong repulsion, the spin sector is dominated by the density-density interaction. Therefore, the spin polarization is not essential. We consider the high polarization case where $M_{m-1} - M_m \ll N$. In this case, the condition $c \gg B_m$ always holds. Thus it is straightforward to calculate density distributions from

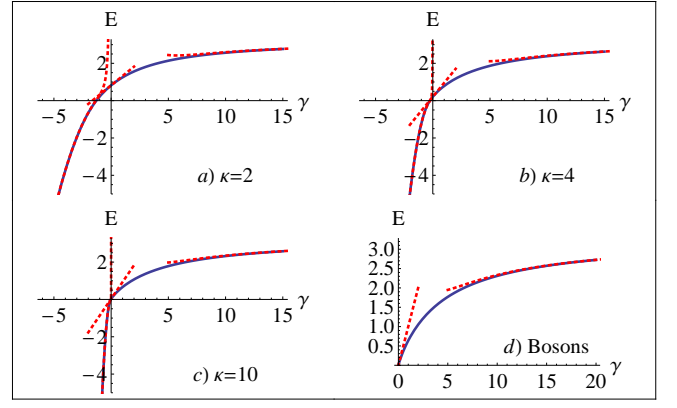


FIG. 2: The ground state energy per length vs $\gamma = cL/N$ in natural units of $2m = \hbar = 1$ for the balanced multicomponent Fermi gas: $\kappa = 2, 4, 10$ and for spinless Bose gas. The solid lines are the numerical solution obtained from the two sets of Fredholm equations. The dashed lines are plotted directly from the asymptotic expansion ground state energies (29), (44) for a repulsive interaction and (55), (59) for an attractive interaction. High precision of the asymptotic expansion ground state energy is seen through out strongly and weakly coupled regimes.

(3) and (4) with a proper Taylor expansion (up to order $O(c^{-3})$), i.e.

$$\begin{aligned} r_0(k) &\approx \frac{1}{2\pi} + \frac{2M_1}{c\pi L} \left[1 - \frac{4k^2}{c^2} \right], \\ r_1(k) &\approx \frac{1}{c\pi L} [2N - M_1 + 2M_2] \\ &\quad - \frac{k^2}{c^3\pi L} [8N - M_1 + 8M_2] - \frac{8E}{c^3\pi}, \\ r_m(k) &\approx \frac{1}{c\pi L} [2M_{m-1} - M_m + 2M_{m+1}] \\ &\quad - \frac{k^2}{c^3\pi L} [8M_{m-1} - M_m + 8M_{m+1}], \quad (30) \end{aligned}$$

where $m = 2, \dots, \kappa$. It is also straightforward to calculate the Fermi boundary B_0 and E from (7) and (8), i.e.

$$\begin{aligned} B_0 &= n\pi \left\{ 1 - \frac{4m_1}{c} + \frac{16m_1^2}{c^2} - \frac{64m_1^3}{c^3} \right. \\ &\quad \left. + \frac{16\pi^2 m_1 n^2}{3c^3} \right\} + O(c^{-4}), \\ E &= \frac{n^3\pi^2}{3} \left\{ 1 - \frac{8m_1}{c} + \frac{48m_1^2}{c^2} - \frac{256m_1^3}{c^3} \right. \\ &\quad \left. + \frac{32\pi^2 m_1 n^2}{5c^3} \right\} + O(c^{-4}), \quad (31) \end{aligned}$$

where $m_1 = M_1/L$ with $M_1 = \sum_{j=1}^{\kappa-1} N^{j+1}$. For the highly polarized case, $M_1 \ll N$, the ground state energy E per unit length, up to $O(c^{-3})$, solely depends on the quantum number M_1 . The spin states are not essential in this strongly repulsive regime due to the freezing of spin

transportation. This result agrees with the energy of the two-component Fermi gas with a strong repulsion, see [16]. It clearly indicates that strong repulsion suppresses the spin effect.

B. Weak repulsion

For the balanced case with weak repulsion, i.e., $cL/N \ll 1$, all B_m with $m \geq 1$ tend to infinity, so the Fredholm equations (6) become

$$\begin{aligned} r_0(k) &= \beta_0 + \int_{-\infty}^{\infty} K_1(k-\lambda)r_1(\lambda)d\lambda, \\ r_m(k) &= \beta_0 - \int_{-\infty}^{\infty} K_m(k-\lambda)r_{0\text{out}}(\lambda)d\lambda \\ &\quad + \int_{-\infty}^{\infty} K_1(k-\lambda)r_{m+1}(\lambda)d\lambda, \end{aligned} \quad (32)$$

where $1 \leq m \leq \kappa - 1$. By iteration with $r_1, r_2, \dots, r_{\kappa-1}$, the first equation of (32) becomes

$$r_0(k) = \kappa\beta_0 - \sum_{s=1}^{\kappa-1} \int_{-\infty}^{\infty} K_{2s}(k-\lambda)r_{0\text{out}}(\lambda)d\lambda. \quad (33)$$

Using a Fourier transformation, we easily prove

$$\begin{aligned} r_{0\text{out}}(k) &= \beta_0 - \int_{-B_0}^{B_0} R(k-\lambda)r_{0\text{in}}(\lambda)d\lambda, \\ R(k) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik\omega} \left[\sum_{s=0}^{\kappa-1} e^{-cs|\omega|} \right]^{-1} d\omega. \end{aligned} \quad (34)$$

Here we see that $R(k) = O(c)$. Substituting the leading term of $r_{0\text{out}}(k) = \beta_0$ into (33), we further obtain

$$\begin{aligned} r_0(k) &= \frac{\kappa}{2\pi} - \frac{1}{2\pi^2} \sum_{s=1}^{\kappa-1} \left\{ \arctan\left(\frac{cs}{B_0-k}\right) \right. \\ &\quad \left. + \arctan\left(\frac{cs}{B_0+k}\right) \right\} + O(c^2) \end{aligned} \quad (35)$$

for the region $|k| \leq B_0$. The energy can be calculated from the equations (7) and (8) with the distribution function (35)

$$E = \frac{\pi^2 n^3}{3\kappa^2} + c(\kappa-1)n^2/\kappa + O(c^2). \quad (36)$$

When $\kappa = 2$ the energy (36) reduces to the result given in [16]. This result clearly indicates a mean field effect among the balanced κ -component weakly interacting fermions. The kinetic energy part in (36) vanishes as $\kappa \rightarrow \infty$. The energy (36) as $\kappa \rightarrow \infty$ is the same as for spinless Bosons with a weak repulsion.

For the imbalanced case with weak repulsion, we assume $B_{m-1} > B_m$ with $1 \leq m \leq \kappa - 1$. The calculation of the ground state energy is very complicated because

the κ integral equations are coupled with each other. We have to separate the integration intervals case by case. Using the Fredholm equations (6), we calculate the following integral in different regions. We denote the integral $I_m = \int_{-B_m}^{B_m} r_m(\lambda)K_\ell(k-\lambda)d\lambda$. For the region $|k| > B_m$, we find

$$I_m = \frac{\ell c B_m}{2\pi^2(k^2 - B_m^2)} + \sum_{s=m+1}^{\kappa-1} \frac{c B_s}{2\pi^2(k^2 - B_s^2)} + O(c^2) \quad (37)$$

After a lengthy calculation, we calculate the integral for the region $|k| \leq B_m$

$$\begin{aligned} I_m &= \beta_0 - \frac{\ell c B_m}{2\pi^2(B_m^2 - k^2)} - \sum_{s=0}^{m-1} \frac{(\ell - m + s + 1)c B_s}{2\pi^2(B_s^2 - k^2)} \\ &\quad + \int_{-B_{m+1}}^{B_{m+1}} r_{m+1}(\lambda)K_{\ell+1}(k-\lambda)d\lambda + O(c^2). \end{aligned} \quad (38)$$

For $B_p < |k| \leq B_{p-1}$ with $p > m$, we further calculate

$$\begin{aligned} I_m &= (p-m)\beta_0 - \sum_{s=0}^{m-1} \frac{(p-m)(\ell+s-m+1)c B_s}{2\pi^2(B_s^2 - k^2)} \\ &\quad - \sum_{s=m}^{p-1} \frac{[(p-s)(\ell+s-m+1)-1]c B_s}{2\pi^2(B_s^2 - k^2)} \\ &\quad + \frac{(\ell+p-m)c B_p}{2\pi^2(k^2 - B_p^2)} + \sum_{s=p+1}^{\kappa-1} \frac{c B_s}{2\pi^2(k^2 - B_s^2)} + O(c^2). \end{aligned} \quad (39)$$

With the help of these formulas, we are able to evaluate the order of $r_m(k)$ in the Fredholm equations (6)

$$\begin{cases} r_m(k) = O(c) & \text{for } |k| > B_{m-1} \\ r_m(k) = \beta_0 + O(c) & \text{for } B_{m+1} < |k| \leq B_{m-1} \\ r_m(k) = (p-m)\beta_0 + O(c) & \text{for } B_p < |k| \leq B_{p-1} \\ & p > m+1 \end{cases} \quad (40)$$

From (7), we obtain the integration boundaries via

$$\begin{aligned} \frac{M_m}{L} &= \frac{B_m}{\pi} + \frac{M_{m+1}}{L} + \frac{c}{2\pi^2} \left\{ \sum_{s=0}^{m-1} \ln \left| \frac{B_s - B_m}{B_s + B_m} \right| \right. \\ &\quad \left. + \sum_{s=m+1}^{\kappa-1} \ln \left| \frac{B_m - B_s}{B_m + B_s} \right| \right\} + O(c^2), \end{aligned} \quad (41)$$

where $0 \leq m \leq \kappa - 1$. In order to calculate the ground state energy, we also need a lengthy calculation of the integral

$$\begin{aligned} \int_{-B_m}^{B_m} \lambda^2 r_m(\lambda)d\lambda &= \frac{B_m^3}{3\pi} + \frac{c B_m}{\pi^2} \left[\sum_{s=0}^{m-1} B_s \right. \\ &\quad \left. + \sum_{s=m+1}^{\kappa-1} B_s \right] + \frac{c}{2\pi^2} \left\{ \sum_{s=0}^{m-1} B_s^2 \ln \left| \frac{B_s - B_m}{B_s + B_m} \right| \right. \\ &\quad \left. + \sum_{s=m+1}^{\kappa-1} B_s^2 \ln \left| \frac{B_m - B_s}{B_m + B_s} \right| \right\} \\ &\quad + \int_{-B_{m+1}}^{B_{m+1}} \lambda^2 r_{m+1}(\lambda)d\lambda + O(c^2). \end{aligned} \quad (42)$$

Using (8) and (42), the ground state energy E per unit length is given by

$$E = \sum_{m=0}^{\kappa-1} \frac{B_m^3}{3\pi} + \frac{c}{2\pi^2} \sum_{m=0}^{\kappa-2} \sum_{r=m+1}^{\kappa-1} [4B_m B_r + (B_m^2 + B_r^2) \ln \left| \frac{B_m - B_r}{B_m + B_r} \right|] + O(c^2). \quad (43)$$

Substituting (41) into (43), we finally obtain the ground state energy of the κ -component Fermi gas with weak repulsion

$$E = \frac{1}{3} \sum_{i=1}^{\kappa-1} (m_{i-1} - m_i)^3 \pi^2 + 2c \sum_{i=1}^{\kappa-1} \sum_{j=i+1}^{\kappa} (m_{i-1} - m_i)(m_{j-1} - m_j) + O(c^2), \quad (44)$$

with $m_0 = n$ and $m_\kappa = 0$. Here the linear density $n = N/L$ and the quantum numbers $m_i = M_i/L$. If we introduce polarization $p_i = N^i/L$ with $i = 1, 2, \dots, \kappa-1$, where N^i is the number of fermions in the i th level. Thus we have a simple form of the ground state energy per length

$$E = \frac{1}{3} \sum_{i=1}^{\kappa} p_i^3 \pi^2 + 2c \sum_{i=1}^{\kappa-1} \sum_{j=i+1}^{\kappa} p_i p_j + O(c^2). \quad (45)$$

The first part is the kinetic energy of the κ -component fermions whereas the second parts is the interaction energy. This result is valid for arbitrary spin imbalance in the weakly repulsive regime. This result is in a good agreement with the numerical calculation, see Fig. 2. For the balanced case, i.e. $M_m - M_{m+1} = N/\kappa$, (44) reduces to the energy (36). It is interesting to note that the ground state energy (45) presents a mean field theory of the two-body s-wave scattering physics.

C. Weak attraction

In weakly attractive coupling regime, the Fredholm equations give the distribution functions of clusters of different sizes. For weak attraction, i.e., $|c|L/N \ll 1$, the two sets of the Fredholm equations for repulsive and attractive regimes preserve the symmetry (16). Therefore, the calculation of the ground state energy E per unit length for weak attraction is similar to that for weak repulsion. For the balanced case with a weak attraction, $N_m = 0$ for $m = 1, \dots, \kappa-1$ and $N_\kappa = N/\kappa$. Thus, we have the condition $Q_m = 0$ for all Fermi points except Q_κ . In this case, the ground state is a spin singlet state. The Fredholm equations (10) for the spin neutral bound state of a κ -atom becomes

$$\rho_\kappa(\lambda) = \kappa\beta_0 + \sum_{s=1}^{\kappa-1} \int_{-Q_\kappa}^{Q_\kappa} K_{2s}(\lambda - \Lambda) \rho_\kappa(\Lambda) d\Lambda. \quad (46)$$

Under the mapping (16), the Fredholm equations (46) for weak attraction map to (33) for weak repulsion. In order to estimate the contribution of density distribution $\rho_{\kappa_{\text{in}}}$, we take the Fourier transformation of (46) and then prove that

$$\rho_{\kappa_{\text{in}}}(\lambda) = \beta_0 - \int_{Q_\kappa}^{Q_\kappa} T(\lambda - \Lambda) \rho_{\kappa_{\text{out}}}(\Lambda) d\Lambda, \\ T(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega\lambda} \left[\sum_{s=0}^{\kappa-1} e^{-s|c\omega|} \right]^{-1} d\omega,$$

where $T(\lambda) = O(c)$. For $|\lambda| \leq Q_\kappa$, we thus have $\rho_\kappa(\lambda) = \rho_{\kappa_{\text{in}}}(\lambda) = \beta_0 + O(c)$. Substituting this leading order into (46), we have

$$\frac{n}{\kappa} = \int_{-Q_\kappa}^{Q_\kappa} \rho_\kappa(\lambda) d\lambda = \frac{Q_\kappa}{\pi} + \frac{(\kappa-1)|c|}{2\pi^2} + \sum_{s=1}^{\kappa-1} \frac{s|c|}{2\kappa\pi^2} \ln \left(\frac{s^2|c|^2 + 4Q_\kappa^2}{s^2|c|^2} \right) + O(c^2). \quad (47)$$

Similarly, from (12), we calculate the ground state energy of the balanced gas

$$E = \frac{\pi^2 n^3}{3\kappa^2} - \frac{|c|(\kappa-1)n^2}{\kappa} + O(c^2). \quad (48)$$

We see that the energy of the balanced gas with weak attraction (48) and with weak repulsion (36) continuously connect at $c \rightarrow 0$. It is also seen that (48) with $\kappa = 2$ is consistent with the result given in [16].

For the imbalanced case, the ground state has cluster states of different sizes. Therefore we assume an ansatz in order to calculate the integration boundaries for each of the Fermi seas, i.e. $Q_m > Q_{m+1}$ with $1 \leq m \leq \kappa-1$. Similarly, we should calculate the integrals $I_m = \frac{1}{2\pi} \int_{-Q_m}^{Q_m} \frac{m|c|r_m(k)dk}{(mc/2)^2 + (\lambda-k)^2}$ for different regions. After some algebra, we find

$$I_m = \frac{\ell|c|Q_m}{2\pi^2(\lambda^2 - Q_m^2)} - \sum_{s=m+1}^{\kappa} \frac{|c|Q_s}{2\pi^2(\lambda^2 - Q_s^2)} + O(c^2) \quad (49)$$

for the region $|\lambda| > Q_m$ and

$$I_m = \beta_0 - \frac{\ell|c|Q_m}{2\pi^2(Q_m^2 - \lambda^2)} + \frac{\ell|c|Q_{m-1}}{2\pi^2(Q_{m-1}^2 - \lambda^2)} + \sum_{s=1}^{m-2} \frac{|c|Q_s}{2\pi^2(Q_s^2 - \lambda^2)} - \sum_{s=m+1}^{\kappa} \int_{-Q_s}^{Q_s} K_{\ell+s-m}(\lambda - \Lambda) \rho_s(\Lambda) d\Lambda + O(c^2). \quad (50)$$

for the region $|\lambda| \leq Q_m$. In the region $Q_{m+1} < |\lambda| \leq Q_m$, we further obtain

$$I_m = \beta_0 + \sum_{s=1}^{m-2} \frac{|c|Q_s}{2\pi^2(Q_s^2 - \lambda^2)} + \frac{\ell|c|Q_{m-1}}{2\pi^2(Q_{m-1}^2 - \lambda^2)} - \frac{\ell|c|Q_m}{2\pi^2(Q_m^2 - \lambda^2)} - \sum_{s=m+1}^{\kappa} \frac{(\ell+1)|c|Q_s}{2\pi^2(\lambda^2 - Q_s^2)} + O(c^2) \quad (51)$$

For $|\lambda| \leq Q_{m+1}$, we find

$$I_m = \frac{(\ell+1)|c|Q_{m+1}}{2\pi^2(Q_{m+1}^2 - \lambda^2)} - \frac{(2\ell+1)|c|Q_m}{2\pi^2(Q_m^2 - \lambda^2)} + \frac{(\ell-1)|c|Q_{m-1}}{2\pi^2(Q_{m-1}^2 - \lambda^2)} + O(c^2). \quad (52)$$

From these equations (49-52), we are able to evaluate the leading order contributions in the density distribution functions, i.e., $\rho_m(\lambda) = (m-p+1)\beta_0 + O(c)$ for $Q_{p+1} < |\lambda| \leq Q_p$ with $p \leq m$ and $\rho_m(\lambda) = O(c)$ for $|\lambda| > Q_m$. From equation (11), we obtain the integration boundaries

$$\frac{N_m}{L} = \frac{Q_m}{\pi} - \sum_{s=m+1}^{\kappa} \frac{N_s}{L} - \frac{|c|}{2\pi^2} \left\{ \sum_{s=1}^{m-1} \ln \left| \frac{Q_s - Q_m}{Q_s + Q_m} \right| + \sum_{s=m+1}^{\kappa} \ln \left| \frac{Q_m - Q_s}{Q_m + Q_s} \right| \right\} + O(c^2). \quad (53)$$

From (12), the ground state energy E per unit length is given by

$$E = \sum_{m=1}^{\kappa} \frac{Q_m^3}{3\pi} - \frac{2|c|}{\pi^2} \sum_{s=1}^{\kappa-1} \sum_{r=s+1}^{\kappa} Q_s Q_r - \frac{|c|}{2\pi^2} \sum_{s=1}^{\kappa-1} \sum_{r=s+1}^{\kappa} (Q_r^2 + Q_s^2) \ln \left| \frac{Q_s - Q_r}{Q_s + Q_r} \right| + O(c^2). \quad (54)$$

Substituting the integration boundaries Q_m from (53) into (54), we thus obtain the ground state energy of the κ -component Fermi gas with a weak attraction

$$E = \frac{1}{3} \sum_{i=1}^{\kappa} p_i^3 \pi^2 - 2|c| \sum_{i=1}^{\kappa-1} \sum_{j=i+1}^{\kappa} p_i p_j, \quad (55)$$

where we introduced polarization $p_i = N^i/L$ with $i = 1, 2, \dots, \kappa$. The first part is the kinetic energy of the κ -component fermions whereas the second part is the interaction energy. This result is in a good agreement with numerical calculation, see Fig.2. For the balanced case, i.e. $N^i = N/\kappa$, (55) reduces to the energy (48). From the energies (45) and (55), we see that the ground state energy of the κ -component gas with arbitrary polarization continuously connects at $c = 0$, precisely speaking, the ground state energy per unit length and its first derivative is continuously connected at $c = 0$. Again, the ground state energy (55) presents a mean field theory of the two-body s-wave scattering physics in weak interacting regimes.

D. Strong Attraction

Universal low temperature behaviours of isospin $S = 1/2, 1, 3/2, \dots, (\kappa-1)/2$ interacting fermions with an attractive interaction in 1D shed light on the nature of

trions, pairing and quantum phase transitions. The existence of these internal degrees of freedom gives rise to some exotic superfluid phases. These models exhibit new quantum phases of matter which are characterized by bound states of different sizes underlying the symmetries. For example, three-component ultracold fermions give rise to a phase transition from a state of trions into the BCS pairing state under external fields. Recently, considerable interest has been paid to the low dimensional strongly interacting fermionic atoms with high spin symmetries.

For a strong attraction, i.e. $|c|L/N \gg 1$, the bound states of different sizes form tightly bound molecules of different sizes with binding energies $\varepsilon_b^{(\ell)} = \ell(\ell^2 - 1)c^2/12$, where $\ell = 2, 3, \dots, \kappa$. In this regime, all the Fermi momenta of the molecules are finite, i.e. $|c| \gg Q_m$ with $m = 1, \dots, \kappa$. Here Q_1 characterizes the Fermi momentum of the single spin-aligned atoms. In the canonical ensemble, external fields or nonlinear Zeeman splittings trigger rich quantum phases and magnetism [12]. A closed form of the ground state energy with polarization is essential to work out phase diagrams and magnetism at zero temperature. In view of the strong attraction condition $|c| \gg Q_m$ and following the method [16], proper Taylor expansions can be carried out for the three supplementary formulas

$$\begin{aligned} \int_{-Q}^Q \bar{K}_\ell(\lambda - \Lambda) d\Lambda &= \frac{4Q}{\pi\ell|c|} - \frac{16Q(Q^2 + 3\lambda^2)}{3\pi\ell^3|c|^3} \\ &+ \frac{64Q(Q^4 + 10Q^2\lambda^2 + 5\lambda^4)}{5\pi\ell^5|c|^5} + O(c^{-7}), \\ \int_{-Q}^Q (\lambda - \Lambda) \bar{K}_\ell(\lambda - \Lambda) d\Lambda &= -\frac{4Q\lambda}{\pi\ell|c|} + \frac{16Q\lambda(Q^2 + \lambda^2)}{\pi\ell^3|c|^3} + O(c^{-5}), \\ \int_{-Q}^Q \Lambda^2 \bar{K}_\ell(\lambda - \Lambda) d\Lambda &= \frac{4Q^3}{3\pi\ell|c|} - \frac{16Q^3(3Q^2 + 5\lambda^2)}{15\pi\ell^3|c|^3} + O(c^{-5}). \end{aligned}$$

In the above calculation, we denote the kernel function $\bar{K}_\ell(k) = \frac{1}{2\pi} \frac{\ell|c|}{(\ell c/2)^2 + k^2}$. The following calculations are valid for arbitrary polarization. For a strong attraction, all terms in the Taylor expansion of the Fredholm equations are converged well, see the method proposed in [16]. From the Fredholm equations (10) and the boundary conditions (11), it is straightforward to obtain the following expression (up to the order $O(c^{-3})$)

$$\begin{aligned} \frac{N_m}{L} &\approx \frac{mQ_m}{\pi} \left\{ 1 - \frac{4}{mL|c|} F_m + \frac{16\pi^2}{3m^3L^3|c|^3} G_m \right\}, \\ \int_{-Q_m}^{Q_m} \Lambda^2 \rho_m(\Lambda) d\Lambda &\approx \frac{mQ_m^3}{3\pi} \left\{ 1 - \frac{4}{mL|c|} F_m \right. \\ &\left. + \frac{16\pi^2}{15m^3L^3|c|^3} \bar{G}_m \right\}, \end{aligned} \quad (56)$$

where we denoted the functions

$$\begin{aligned}
F_m &= \sum_{s=1}^{m-1} \sum_{r=1}^s \frac{N_s}{2r-s+m-2} + \sum_{r=1}^{m-1} \frac{N_m}{2r} \\
&\quad + \sum_{s=m+1}^{\kappa} \sum_{r=1}^m \frac{N_s}{2r+s-m-2}, \\
G_m &= \sum_{s=1}^{m-1} \sum_{r=1}^s \frac{s^2 N_m^2 N_s + m^2 N_s^3}{s^2(2r-s+m-2)^3} + \sum_{r=1}^{m-1} \frac{2N_m^3}{(2r)^3} \\
&\quad + \sum_{s=m+1}^{\kappa} \sum_{r=1}^m \frac{s^2 N_m^2 N_s + m^2 N_s^3}{s^2(2r+s-m-2)^3}, \\
\bar{G}_m &= \sum_{s=1}^{m-1} \sum_{r=1}^s \frac{9s^2 N_m^2 N_s + 5m^2 N_s^3}{s^2(2r-s+m-2)^3} + \sum_{r=1}^{m-1} \frac{14N_m^3}{(2r)^3} \\
&\quad + \sum_{s=m+1}^{\kappa} \sum_{r=1}^m \frac{9s^2 N_m^2 N_s + 5m^2 N_s^3}{s^2(2r+s-m-2)^3}.
\end{aligned}$$

Here N_i with $i = 1, 2, \dots$, are the numbers of the cluster state of i -atom.

After a lengthy calculation, we obtain explicit forms of the Fermi momenta and the energies of the molecules of different sizes for a strong attraction

$$\begin{aligned}
Q_m &\approx \frac{N_{m\pi}}{mL} \left\{ 1 + \frac{4}{mL|c|} F_m + \frac{16}{m^2 L^2 |c|^2} F_m^2 \right. \\
&\quad \left. + \frac{16}{3m^3 L^3 |c|^3} [12F_m^3 - G_m \pi^2] \right\}, \quad (57)
\end{aligned}$$

$$\begin{aligned}
E_m &\approx \frac{\pi^2 N_m^3}{3mL^3} \left\{ 1 + \frac{8}{mL|c|} F_m \right. \\
&\quad + \frac{48}{m^2 L^2 |c|^2} F_m^2 + \frac{256}{m^3 L^3 |c|^3} F_m^3 \\
&\quad \left. + \frac{16\pi^2}{m^3 L^3 |c|^3} [-G_m + \bar{G}_m/15] \right\}. \quad (58)
\end{aligned}$$

Thus the ground state energy of the gas with arbitrary polarization per length in strong attractive regime is given by

$$E = \sum_{\ell=1}^{\kappa} (E_{\ell} - n_{\ell} \varepsilon_b^{(\ell)}), \quad (59)$$

where the binding energy of the molecule state of ℓ -atom is given by $\varepsilon_b^{(\ell)} = \ell(\ell^2 - 1)c^2/12$. For $\kappa = 2$, it covers the result obtained for the two-component Fermi gas given in [16]. From the discrete BA equations, one of the author and coworkers [12] derived the ground state energy of κ -component strongly attractive Fermi gas for up to the order $O(1/c^2)$. Here we obtained a more accurate ground state energy (58) of the κ -component gas with arbitrary polarization from the analytical study of the Fredholm equations. We noticed that the ground state of strongly attractive κ -component Fermi gases can be effectively described by a super Tonks-Girardeau gaslike state via a proper mapping [21]. The explicit

form of the ground state energy can be used to study magnetism and full phase diagrams of the model from the relations (13) and (14) in a straight forward way. The effective chemical potentials may be calculated from $\mu_m = \partial(\sum_{\ell=1}^{\kappa} E_{\ell}) / (m \partial n_m)$, where $n_m = N_m/L$. From these effective fields, the phase diagrams can be determined by the energy-field transfer relations (14), see recent study of the three-component attractive Fermi gas [13, 22].

IV. LOCAL PAIR CORRELATIONS

There has been a considerable interest in studying universal nature of interacting fermions. Remarkably, Tan [23] showed that the momentum distribution exhibits universal \mathcal{C}/k^4 decay as the momentum tends to infinity. Here the constant \mathcal{C} is called universal contact that measures the probability of two fermions with opposite spin at the same position. For 1D two-component Fermi gas [24], the universal contact is obtained by calculating the change of the interacting energy with respect to interaction strength by Hellman-Feynman theorem, i.e. $\mathcal{C} = \frac{4}{a_{1D}^2} n_{\uparrow} n_{\downarrow} g_{\uparrow, \downarrow}^{(2)}(0)$. The local pair correlation $g_{\uparrow, \downarrow}^{(2)}$ are accessible via exact Bethe ansatz solution through the relation $g_{\uparrow, \downarrow}^{(2)}(0) = \frac{1}{2n_{\uparrow} n_{\downarrow}} \partial E / \partial c$. Here E is the ground state energy per length.

In a similar way, for 1D κ -component Fermi gas, there exists a 1D analog of the Tan adiabatic theorem where the universal contact is given by the local pair correlations for two fermions with different spin states. The two-body local pair correlation function is similar to the calculation of the expectation value of the four-operator term in the second quantized Hamiltonian. For a homogenous and balanced κ -component Fermi gas, the local pair correlation function is given by

$$g_{\sigma, \sigma'}^{(2)}(0) = \frac{\kappa}{(\kappa - 1)n^2} \frac{\partial E}{\partial c} \quad (60)$$

with $\kappa > 1$. Where E is the ground state energy per length. From the asymptotic expansion result of the ground state energy of the balanced Fermi gas obtained in the above section, we easily find $g_{\sigma, \sigma'}^{(2)}(0) \rightarrow 1$ as $|c| \rightarrow 0$. From the ground state energy (29) of the balanced gas with a strong repulsion, we have the local pair correlation

$$g_{\sigma, \sigma'}^{(2)}(0) = \frac{4\kappa\pi}{3(\kappa - 1)\gamma^2} \left[Z_1 - \frac{6Z_1^2}{\gamma} + \frac{24}{\gamma^2} \left(Z_1^3 - \frac{Z_3\pi^2}{15} \right) \right] \quad (61)$$

This local pair correlation reduces to the one for the spinless Bose gas [18, 25] as $\kappa \rightarrow \infty$.

For strong attractive interaction, we obtain the ground state energy of the balanced gas from the result (58)

$$\begin{aligned}
E &= \frac{\pi^2 n^3}{3\kappa^4} \left(1 + \frac{4A_{\kappa}}{\kappa^2 |\gamma|} + \frac{12A_{\kappa}^2}{\kappa^4 \gamma^2} + \frac{32A_{\kappa}^3}{\kappa^6 |\gamma|^3} \right. \\
&\quad \left. - \frac{32B_{\kappa}}{15\kappa^6 \gamma^3} \right) - n_{\kappa} \varepsilon_b^{(\kappa)} + O(1/\gamma^4), \quad (62)
\end{aligned}$$

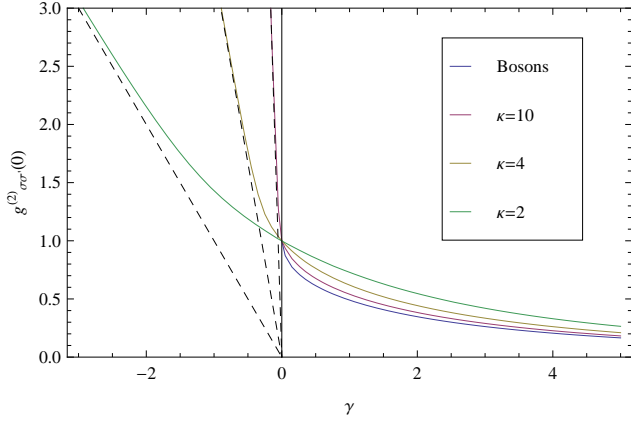


FIG. 3: The local pair correlation $g_{\sigma,\sigma'}^{(2)}(0)$ vs $\gamma = cL/N$ in natural units for the balanced multicomponent Fermi gas: $\kappa = 2, 4, 10$ and for spinless Bose gas. The solid lines are the numerical solutions obtained from the two sets of Fredholm equations. The dashed lines are the asymptotic limits obtained from the first term in (63). All local correlations pass the point $g_{\sigma,\sigma'}^{(2)}(0) = 1$ at vanishing interaction strength. However, the local pair correlation diverges for the attractive Bose gas.

where $A_\kappa = \sum_{r=1}^{\kappa-1} 1/r$ and $B_\kappa = \sum_{r=1}^{\kappa-1} 1/r^3$. From the relation (60), we obtained the local pair correlation for two fermions with different spin states in a strong attractive regime

$$g^{(2)} = \frac{\kappa(\kappa+1)|\gamma|}{6} + \frac{4\pi^2}{3\kappa^5(\kappa-1)\gamma^2} \left(A_\kappa + \frac{6A_\kappa^2}{\kappa^2|\gamma|} + \frac{24}{\kappa^4\gamma^2} \left(A_\kappa^3 - \frac{B_\kappa}{15} \right) \right) + O(1/\gamma^4). \quad (63)$$

We see that the local pair correlation becomes divergent as $\kappa \rightarrow \infty$, i.e. goes to the limit for the attractive bosons. In Fig.3, we present the numerical solution of the local pair correlation for two fermions with different spin states in the multicomponent Fermi gas. In the whole interacting regime, the local pair correlations for the balanced κ -component Fermi gas tends to the limit value for the spinless bosons as $\kappa \rightarrow \infty$. High precision of these asymptotic expansion local pair correlations is seen in strong and weak interaction regions, see Fig.4.

V. CONCLUSION

We have analytically studied the Fredholm equations for the 1D κ -component fermions with repulsive and attractive δ -function interactions in four regimes: a) strong repulsion; B) weak repulsion; C) weak attraction and D) strong attraction. Solving two sets of the Fredholm equations (4) and (10), we have obtained the first few terms of the asymptotic expansion for the density distribution functions and the ground state energy E per unit length

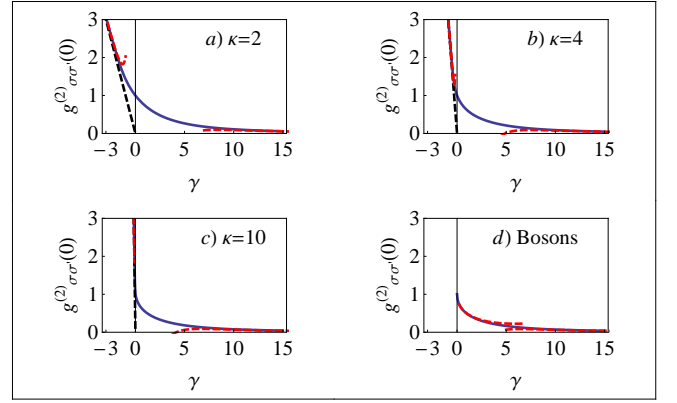


FIG. 4: The local pair correlation $g_{\sigma,\sigma'}^{(2)}(0)$ vs $\gamma = cL/N$ in natural units for the balanced multicomponent Fermi gas: $\kappa = 2, 4, 10$ and for spinless Bose gas. The solid lines are the numerical solutions obtained from the two sets of Fredholm equations. The dashed lines are the asymptotic expansion result obtained from (61) and (63).

for both balanced and unbalanced cases in these regimes. We summarize our main result as following.

A) For the strong repulsive regime, the ground state energy of the balanced κ -component gas has been given in (29) up to the order of $1/c^3$. It presents a universal structure in terms of the parameters Z_1 and Z_3 (27) that characterize quantum statistical and dynamical effects of the model. It also confirms Yang and You's result [19] that the energy per particle as $\kappa \rightarrow \infty$ is the same as for spinless Bosons. In this regime, we have also obtained the ground state energy for the highly imbalanced case (31). It clearly indicates that the internal spin effect is strongly suppressed in this regime, i.e. the energy only depends on the quantum number M_1 .

B) For the weak repulsive regime, we have calculated the ground state energy E (36) for the balanced case and (44) for the imbalanced case. The ground state energy (44) presents a simple mean field theory of the two-body s-wave scattering physics. We see that the energy (36) for the balanced κ -component Fermi gas is the same as that for the spinless Bose gas as $\kappa \rightarrow \infty$, consistent with the result in [19].

C) For the weak attractive regime, we have also calculated the ground state energy E (48) for the balanced case and (55) for the imbalanced case. We have found that the two sets of the Fredholm equations for the model with repulsive and attractive interactions preserves a particular mapping (16). But they do not preserve the density mapping found for the two-component Fermi gas [16]. Our result shows that the ground state energy of the model with arbitrary spin state is continuous at $c \rightarrow 0$, as is the first derivative.

D) For the strong attractive regime, the highly accurate ground state energy (up to $O(c^{-3})$) of the κ -component with arbitrary spin polarization has been given in (58). The ground state of the largest cluster

state can be effectively described by the gas like state in the super Tonks-Girardeau gas of hard core bosons. From the result of ground state energy of the strongly attractive fermions, we can study magnetism and identify full phase diagrams of the gas with $SU(\kappa)$ symmetry driven by the external fields.

Furthermore, for the balanced case, we have obtained the first few terms of the asymptotic expansion for the local pair correlations of two fermion with different spin states in the four regimes. A numerical solution of the local pair correlation function has been obtained in the whole interacting regime. These results we obtained provide insight into understanding quantum statistical and dynamical effects in 1D interacting fermions with high spin symmetries. The explicit forms of the ground state

energy of the multicomponent Fermi gas with polarization give further applications to the study of quantum phase transitions, magnetism, and phase diagrams, which provides useful guides for future experiments with ultracold atomic Fermi gases with high symmetries.

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